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Analysis of dynamic instability for arbitrarily laminated skew plates

G.Y. Wu[a,*], Y.S. Shih[b]

[a]*Department of Fire Science, Central Police University, 56, Shu-jen Rd., Ta-kang, Kwei-shan, Tao-yuan, 333, Taiwan, ROC*
[b]*Department of Mechanical Engineering, Chung-Yuan Christian University, Chung-Li, 320, Taiwan, ROC*

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Abstract

The dynamic instability and nonlinear response of rectangular and skew laminated plates subjected to periodic in-plane load are studied. Based on von Karman plate theory, the large amplitude dynamic equations of thin laminated plates are derived by applying the approach of generalized double Fourier series. On the assumed mode shape, the governing equations are reduced to the Mathieu equation using Galerkin's method. The incremental harmonic balance (IHB) method is applied to solve the nonlinear temporal equation of motion, and the region of dynamic instability is determined in this work. Calculations are carried out for isotropic, angle-ply and arbitrarily laminated plates under two cases of boundary conditions. The principal region of dynamic instability associated with the effect of the stacking sequence of lamination and the skew angle of plate are also investigated and discussed. The results obtained indicated that the instability behavior of the system is determined by the several parameters, such as the boundary condition, number of the layers, stacking sequence, in-plane load, aspect ratio, amplitude and the skew angle of plate.

1. Introduction

The dynamic instability of laminated plates has received considerable attention in recent years. As is well known, when a flat plate sustains a period in-plane force, it may become laterally

*Corresponding author. Tel.: +886 3 318 5326; fax: +886 3 328 1114.
E-mail address: una210@sun4.cpu.edu.tw (G.Y. Wu).

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unstable over certain regions of the parameter space such as the excitating frequency, the in-plane load or the amplitude of system, etc., and this phenomenon is referred to as parametric or dynamic instability. The dynamic instability of rectangular plates under various in-plane periodic forces has been presented by Bolotin [1] and Evan-Iwanowski [2].

The nonlinear dynamic equations of motion resulting from von Karman's theory have been formulated for laminated anisotropic plates by Whitney and Leissa [3]. Kraicinovic and Herrmann [4] solved the dynamic stability problem of an isotropic plate using an integral equation technique. The dynamic stability of simply supported antisymmetric angle-ply rectangular plates has been presented by Bert and Birman [5]. Reddy [6], and Chen and Yang [7], and Srinivasan and Chellapandi [8] used the finite element model to study the free vibration and the dynamic instability of rectangular laminated composite plates. Ostiguy and Evan-Iwanowski [9], Nguney and Ostiguy [10,11] considered the influence of the aspect ratio and boundary conditions on the dynamic instability and nonlinear response of rectangular plates theoretically and experimentally. Duffield and Willems [12] presented an analytical and experimental investigation parametric instability of a stiffened rectangular plate.

For skew plates, Sathyamoorthy and Pandalai [13,14] have studied the large amplitude flexural vibrations of thin elastic orthotropic skew plates. Liew and Lam [15] have proposed a set of two-dimensional orthogonal plate functions to obtain solutions for flexural vibration of skew plates. The dynamic stability of skew stiffened laminated composite plates investigated by Merritt and Willems [16]. Kamal and Durvasula [17] applied a double series to carry out the stability analysis of composite skew plates based on small deflection thin plate theory. Liao and Cheng [18] used finite element method to analyze the dynamic stability of stiffened skew plates. Reddy and Palaninathan [19] have studied the free vibration analysis of laminated composite skew plates and the effects of skew angle, number of layers and fiber orientation angle.

In this study, solutions to large amplitude dynamic instability of skew thin laminated plates subjected to both static and periodic in-plane force applied along two opposite edges are determined. Using the Galerkin's method, the governing equations are reduced to a time-dependent Mathieu equation. The influence of boundary conditions on the stability characteristics

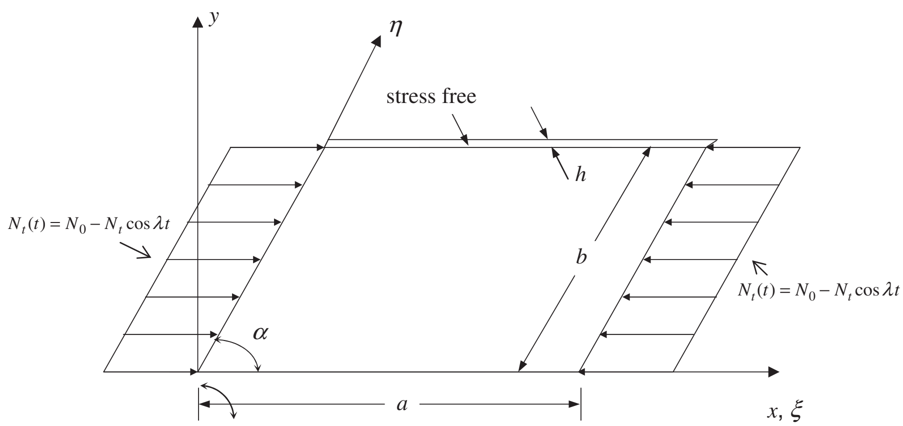

Fig. 1. Plate and load configuration.

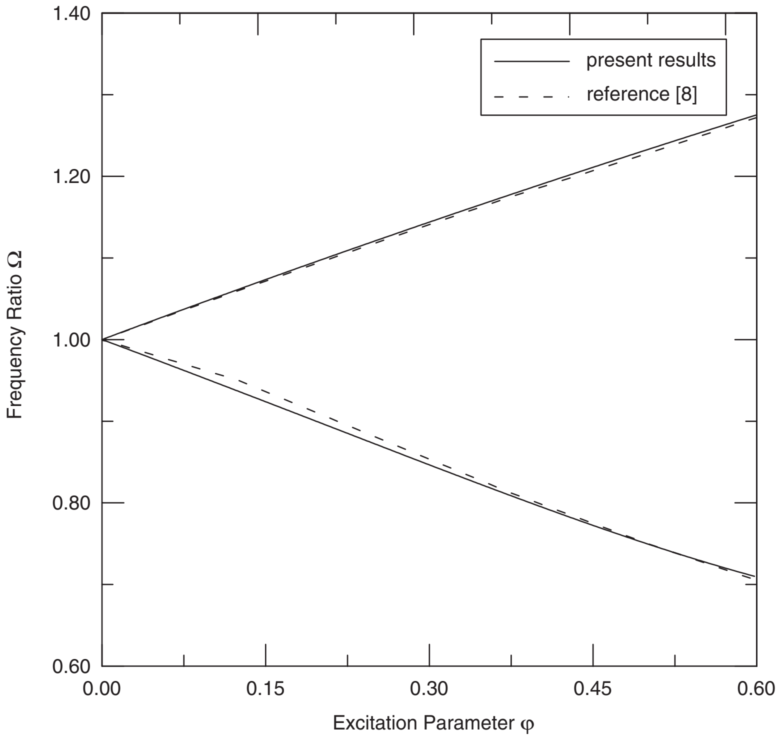

Fig. 2. The comparison of the principal instability regions for the rectangular isotropic clamped plates: $a_1 = 0.01$, $b_1 = 0.01$, $E_1/E_2 = 1.0$, $v = 0.3$.

and the dynamic instability of plates are investigated by the incremental harmonic balance (IHB) method. The results of regions of dynamic stability for isotropic, antisymmetric angle-ply and arbitrarily laminated rectangular and skew plate are provided.

2. Governing equations

Equations of motion for generally laminated plates based on von Karman's plate theory are given by Chia [20], and are shown as follows:

$$N_{x,x} + N_{xy,y} = 0, \tag{1a}$$

$$N_{y,y} + N_{xy,x} = 0, \tag{1b}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} = \rho h w_{tt}, \tag{1c}$$

$$\varepsilon_x^\circ = u_{,x}^\circ + \tfrac{1}{2} w_{,x}^2, \quad \varepsilon_y^\circ = v_{,y}^\circ + \tfrac{1}{2} w_{,y}^2, \quad \varepsilon_{xy}^\circ = u_{,y}^\circ + v_{,x}^\circ + w_{,x} w_{,y},$$

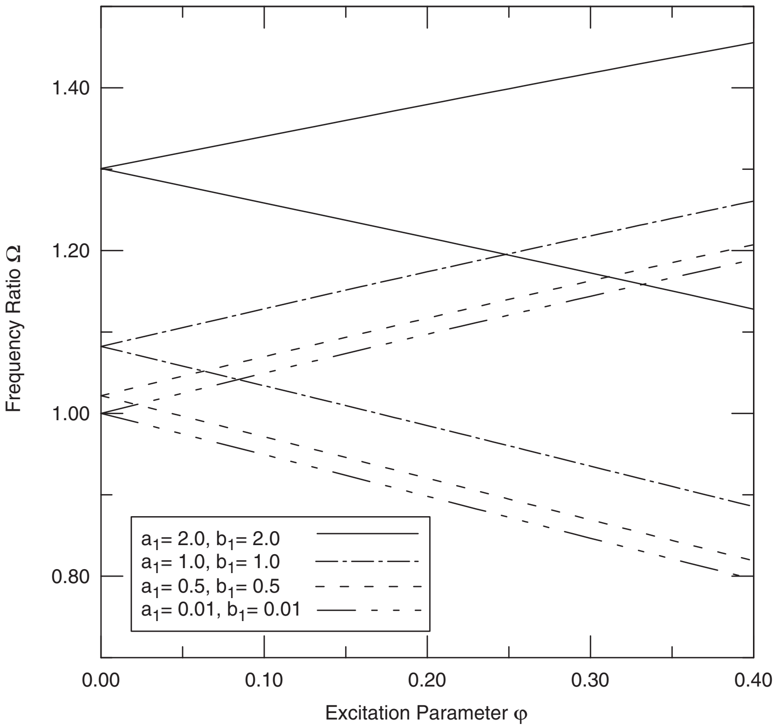

Fig. 3. Parameter instability of the isotropic simply supported rectangular plate.

$$\left\{\begin{matrix}\varepsilon^{\circ}\\ M\end{matrix}\right\} = \begin{bmatrix} A^{*} & B^{*} \\ -(B^{*})^{\mathrm{T}} & D^{*}\end{bmatrix}\left\{\begin{matrix}N\\ K\end{matrix}\right\},$$

$$\{M\} = (M_x, M_y, M_{xy})^{\mathrm{T}} = -[B^{*}]^{\mathrm{T}}\{N\} + [D^{*}]\{K\},$$

$$\{K\}^{\mathrm{T}} = (K_x, K_y, K_{xy}) = -(w_{,xx}, w_{,yy}, 2w_{,xy}),$$

$$[A^{*}] = [A]^{-1},\ \ [B^{*}] = -[A]^{-1}[B], \quad \text{and} \quad [D^{*}] = [D] - [B][A]^{-1}[B] = [D] + [B][B^{*}],$$

where ε_x°, ε_y° and ε_{xy}° are mid-surface strain components in rectangular Cartesian coordinates; u°, v° and w are displacement components in mid-surface in x-, y-, z-directions, respectively; M_x, M_y and M_{xy} are bending and twisting moments; N_x, N_y and N_{xy} are membrane forces; A_{ij}, B_{ij} and D_{ij} are extension stiffness, coupling stiffness and bending stiffness, respectively; K_x, K_y and K_{xy} are curvatures at mid-plane; h is thickness of the plate; ρ is mass density per unit volume of plate.

Equation of compatibility is

$$\varepsilon^{\circ}_{x,yy} + \varepsilon^{\circ}_{y,xx} - \varepsilon^{\circ}_{xy,xy} = w^2_{,xy} - w_{,xx}w_{,yy}. \tag{2}$$

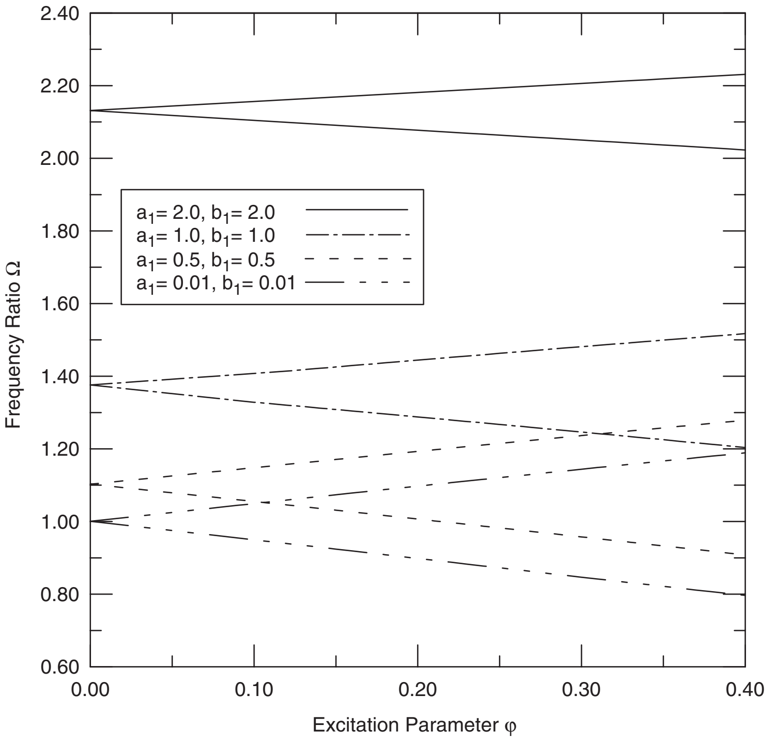

Fig. 4. Parameter instability of the isotropic clamped rectangular plate.

The stress function (ϕ) is defined such that

$$\{N\}^{\mathrm{T}} = (N_x, N_y, N_{xy}) = (\phi_{,yy}, \phi_{,xx}, -\phi_{,xy}). \tag{3}$$

Then, the two coupled governing equations of arbitrarily laminated thin plates based on x–y coordinate are derived as follows:

$$L_2\phi - L_3 w + w_{,xx}w_{,yy} - w_{,xy}^2 = 0, \tag{4}$$

$$\rho h w_{tt} + L_1 w + L_3\phi - L(\phi, w) = 0, \tag{5}$$

where

$$\begin{aligned}
L_1 &= D_{11}^*()_{,xxxx} + 4D_{16}^*()_{,xxxy} + 2(D_{12}^* + 2D_{66}^*)()_{,xxyy} + 4D_{26}^*()_{,xyyy} + D_{22}^*()_{,yyyy},\\
L_2 &= A_{22}^*()_{,xxxx} - 2A_{26}^*()_{,xxxy} + (2A_{12}^* + A_{66}^*)()_{,xxyy} - 2A_{16}^*()_{,xyyy} + A_{11}^*()_{,yyyy},\\
L_3 &= B_{21}^*()_{,xxxx} + (2B_{26}^* - B_{61}^*)()_{,xxxy} + (B_{11}^* + B_{22}^* - 2B_{66}^*)()_{,xxyy}\\
&\quad + (2B_{16}^* - B_{62}^*)()_{,xyyy} + B_{12}^*()_{,yyyy},
\end{aligned}$$

$$L(\phi, w) = \phi_{,yy}w_{,xx} + \phi_{,xx}w_{,yy} - 2\phi_{,xy}w_{,xy}.$$

In skew plate analysis, it is expedient to use the oblique coordinates (ξ, η) instead of the Cartesian coordinates (x, y). The (ξ, η) coordinate system is shown in Fig. 1 and the

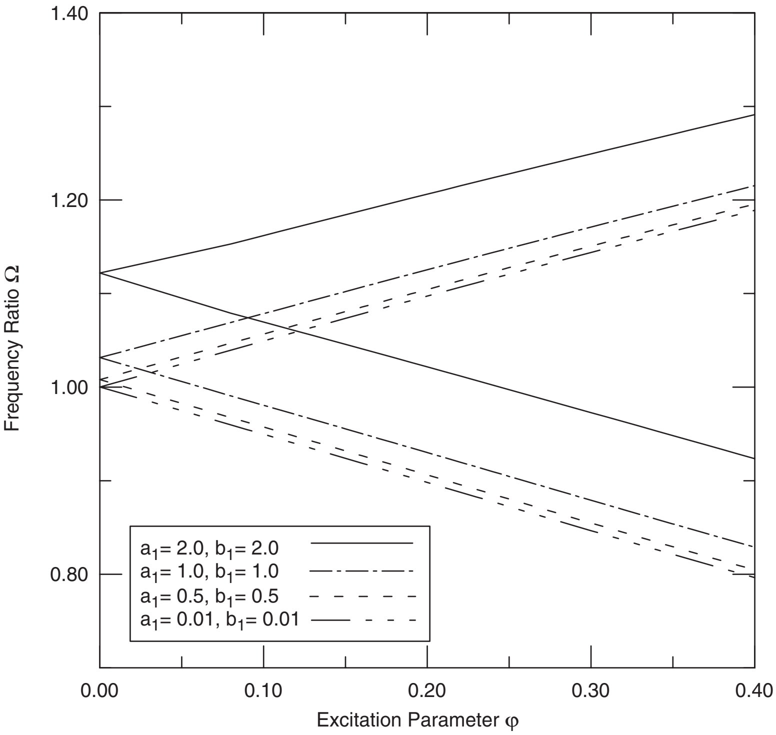

Fig. 5. Parameter instability of the isotropic simply supported skew (α = 60°) plate.

transformations between (x, y) and (ξ, η) coordinate systems are presented as follows:

$$x = \xi + \eta\cos\alpha, \quad \xi = x - y\cot\alpha, \tag{6}$$

$$y = \eta\sin\alpha, \quad \eta = y\csc\alpha, \tag{7}$$

where α is the skew angle of the plate.

Using the differential Chain Rule, Eqs. (4) and (5) can be rewritten in terms of the oblique coordinates (ξ, η) as

$$L_2\phi - L_3 w + \csc^2\alpha(w_{,\eta\eta}w_{,\xi\xi} - w^2_{,\xi\eta}) = 0, \tag{8}$$

$$\rho h w_{,tt} + L_1 w + L_3\phi - L(\phi, w) = 0, \tag{9}$$

where

$$\begin{aligned} L_1 = {} & [D^*_{11} - 4D^*_{16}\cot\alpha + 2(D^*_{12} + 2D^*_{66})\cot^2\alpha - 4D^*_{26}\cot^3\alpha + D^*_{22}\cot^4\alpha]()_{,\xi\xi\xi\xi} \\ & + [2(D^*_{12} + 2D^*_{16})\csc^2\alpha - 12D^*_{26}\cot\alpha\csc^2\alpha + 6D^*_{22}\cot^2\alpha\csc^2\alpha]()_{,\xi\xi\eta\eta} \\ & + [4D^*_{26}\csc^3\alpha - 4D^*_{22}\cot\alpha\csc^3\alpha]()_{,\xi\eta\eta\eta} + D^*_{22}\csc^4\alpha()_{,\eta\eta\eta\eta} \\ & + [4D^*_{16}\csc\alpha - 4\cot\alpha\csc\alpha(D^*_{12} + 2D^*_{66}) + 12D^*_{26}\cot^2\alpha\csc\alpha \\ & - 4D^*_{22}\cot^3\alpha\csc\alpha]()_{,\xi\xi\xi\eta}, \end{aligned}$$

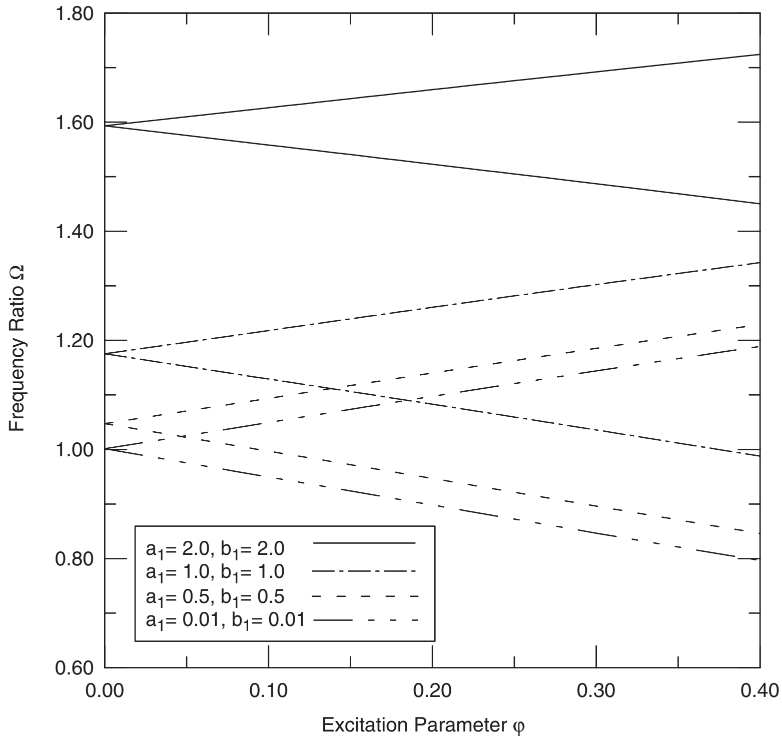

Fig. 6. Parameter instability of the isotropic clamped skew ($\alpha = 60^{\circ}$) plate.

Table 1
The effects of plate ratio (a/b) on dimensionless fundamental frequency for simply supported square angle-ply [45°/−45°/45°/−45°] plates ($a/h = 50$)

a/b	Reddy [6]	Bert and Chen [25]	Present
0.2	9.816	9.507	9.549
0.4	12.280	11.82	11.892
0.6	15.689	15.04	15.164
0.8	19.759	18.89	19.078
1.0	24.343	23.24	23.528
1.2	29.321	28.06	28.489
1.4	34.742	33.37	33.970
1.6	40.653	39.17	39.993
1.8	47.067	—	46.576
2.0	53.989	52.29	53.744

$$
\begin{aligned}
L_2 = {} & [A_{22}^* + 2A_{26}^* \cot\alpha + (2A_{12}^* + A_{66}^*)\cot^2\alpha + 2A_{16}^*\cot^3\alpha + A_{11}^*\cot^4\alpha]()_{,\xi\xi\xi\xi} \\
& + [(2A_{12}^* + A_{66}^*)\csc^2\alpha + 6A_{16}^*\cot\alpha\csc^2\alpha + 6A_{11}^*\cot^2\alpha\csc^2\alpha]()_{,\xi\xi\eta\eta} \\
& - [2A_{16}^*\csc^3\alpha + 4A_{11}^*\cot\alpha\csc^3\alpha]()_{,\xi\eta\eta\eta} + A_{11}^*\csc^4\alpha()_{,\eta\eta\eta\eta}
\end{aligned}
$$

Table 2
The effects of plate ration (a/b) on dimensionless fundamental frequency for simply supported square angle-ply [30°/−30°/30°/−30°] plates ($a/h = 50$)

a/b	Reddy [6]	Present
0.2	13.233	13.053
0.4	14.700	14.380
0.6	16.908	16.467
0.8	19.667	19.098
1.0	22.850	22.158
1.2	26.389	25.578
1.4	30.229	29.326
1.6	34.330	33.387
1.8	38.677	37.759
2.0	43.283	42.441

Table 3
The dimensionless fundamental frequency for the simply supported square angle-ply laminated plate ($a/h = 100$)

[45°/−45°]			[45°/−45°/45°/−45°]			[45°/−45°/…45°/−45°] (8 layers)		
Reddy [6]	Chen and Yang [7]	Present	Reddy [6]	Chen and Yang [7]	Present	Reddy [6]	Chen and Yang [7]	Present
14.618	14.90	14.635	—	23.66	23.528	25.176	25.38	25.267

Table 4
The dimensionless fundamental frequency for the simply supported antisymmetric angle-ply laminated plate

Fiber angle	2 layers		4 layers		6 layers	
	Reddy and Palaninathan [19]	Present	Reddy and Palaninathan [19]	Present	Reddy and Palaninathan [19]	Present
0°	18.806	18.805	18.806	18.805	18.806	18.805
15°	14.646	13.538	19.431	19.276	20.193	20.160
30°	14.204	14.006	22.175	22.158	23.355	23.357
45°	14.638	14.635	23.258	23.528	24.828	24.827
60°	14.204	14.006	22.175	22.158	23.355	23.357
75°	14.646	13.538	19.431	19.276	20.193	20.160
90°	18.806	18.805	18.806	18.805	18.806	18.805

$$
\begin{aligned}
&- [2A_{26}^{*} \csc\alpha + 2(2A_{12}^{*} + A_{66}^{*}) \cot\alpha \csc\alpha + 6A_{16}^{*} \cot^{2}\alpha \csc\alpha \\
&+ 4A_{11}^{*} \cot^{3}\alpha \csc\alpha]()_{,\xi\xi\xi\eta}, \\
L_3 = {}& [B_{21}^{*} - (2B_{26}^{*} - B_{61}^{*}) \cot\alpha + (B_{11}^{*} + B_{22}^{*} - 2B_{66}^{*})\cot^{2}\alpha - (2B_{16}^{*} - B_{62}^{*})\cot^{3}\alpha \\
&+ B_{12}^{*} \cot^{4}\alpha]()_{,\xi\xi\xi\xi} + [(2B_{26}^{*} - B_{61}^{*})\csc\alpha - 2\cot\alpha\csc\alpha(B_{11}^{*} + B_{22}^{*} - 2B_{66}^{*})
\end{aligned}
$$

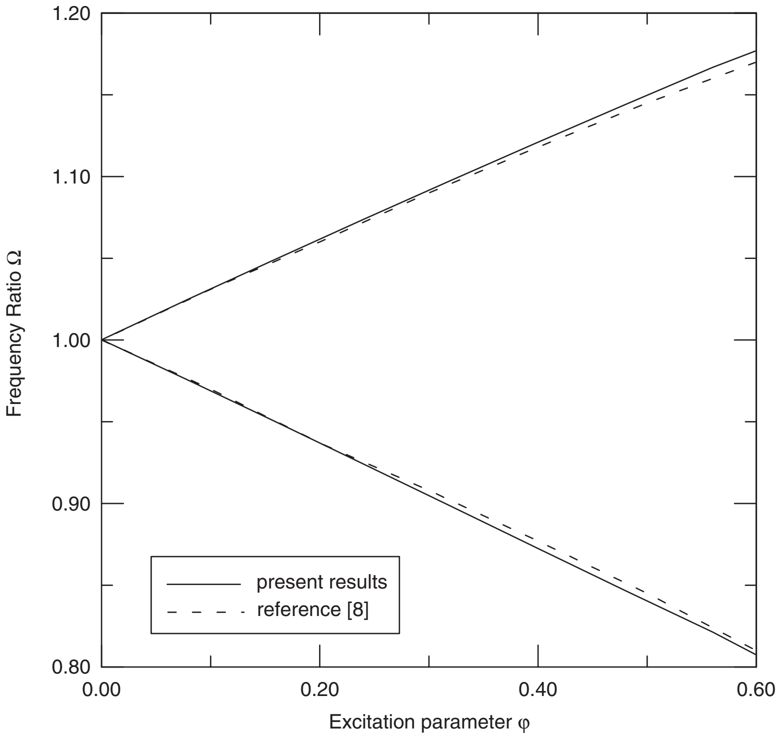

Fig. 7. The comparison of the principal instability regions for the clamped rectangular angle-ply [45°/−45°/45°/−45°] laminated plates: $a_1 = 0.01$, $b_1 = 0.01$, $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = G_{23}/E_2 = 0.5$, $v_{12} = 0.25$.

$$
\begin{aligned}
&+ 3\cot^2\alpha\csc\alpha(2B^*_{16} - B^*_{62}) - 4B^*_{12}\cot^3\alpha\csc\alpha]()_{,\xi\xi\xi\eta} + [(B^*_{11} + B^*_{22} - 2B^*_{66}) \\
&\times\csc^2\alpha - 3\cot\alpha\csc^2\alpha(2B^*_{16} - B^*_{62}) + 6B^*_{12}\cot^2\alpha\csc^2\alpha]()_{,\xi\xi\eta\eta} \\
&+ [(2B^*_{16} - B^*_{62})\csc^3\alpha - 4B^*_{12}\cot\alpha\csc^3\alpha]()_{,\xi\eta\eta\eta} + B^*_{12}\csc^4\alpha()_{,\eta\eta\eta\eta}
\end{aligned}
$$

$$
L(\phi, w) = \csc^2\alpha(\phi_{,\eta\eta}w_{,\xi\xi} + \phi_{,\xi\xi}w_{,\eta\eta} - 2\phi_{,\xi\eta}w_{,\xi\eta}).
$$

3. Analysis

3.1. Mathieu's equation

A laminate plate shown in Fig. 1 is simply supported at all edges (S–S–S–S) or clamped at all edges (C–C–C–C), and subjected to the action of periodic in-plane force uniformly distributed along two opposite edges.

The boundary conditions for all edges simply supported (S–S–S–S) can be written as

$$
\begin{aligned}
&w = 0, \quad M_\xi = 0 \quad \text{at } \xi = 0, a, \\
&w = 0, \quad M_\eta = 0 \quad \text{at } \eta = 0, b.
\end{aligned}
\tag{10}
$$

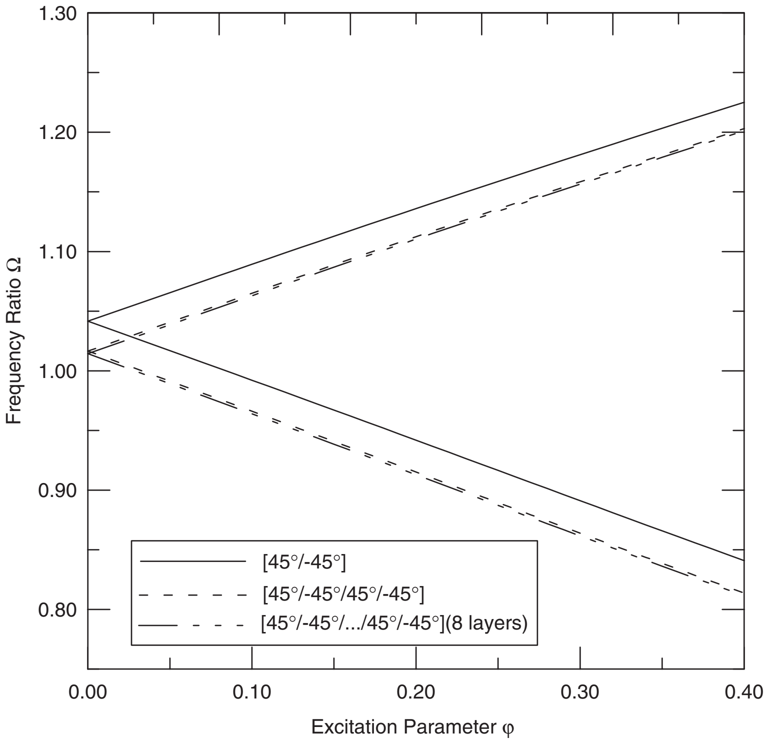

Fig. 8. The effect of the number of layers on the principal instability regions for the simply supported rectangular angle-ply laminated plates.

The boundary conditions for all edges clamped (C–C–C–C) can be written as follows:

$$\begin{aligned} &w = 0, \quad w_{,\eta} = 0 \quad \text{at } \eta = 0, b, \\ &w = 0, \quad w_{,\xi} = 0 \quad \text{at } \xi = 0, a. \end{aligned} \tag{11}$$

A transverse deflection function w satisfying the boundary conditions, is assumed to be

$$w = \sum_m \sum_n hW_{mn}\psi_m(\xi)Z_n(\eta). \tag{12}$$

Substituting Eq. (12) into Eqs. (10) and (11) yields, for a simply supported edge,

$$\psi_m(\xi) = \sin\left(\frac{m\pi\xi}{a}\right) \quad \text{and} \quad Z_n(\eta) = \sin\left(\frac{n\pi\eta}{b}\right) \tag{13}$$

and, for a clamped edge,

$$\psi_m(\xi) = X_m(\xi) \quad \text{and} \quad Z_n(\eta) = Y_n(\eta), \tag{14}$$

where X_i, Y_j are beam eigenfunction given by

$$X_i(\xi) = C_i\left(\cosh\left(\frac{\lambda_i\xi}{a}\right) - \cos\left(\frac{\lambda_i\xi}{a}\right)\right) - \left(\sinh\left(\frac{\lambda_i\xi}{a}\right) - \sin\left(\frac{\lambda_i\xi}{a}\right)\right), \tag{15}$$

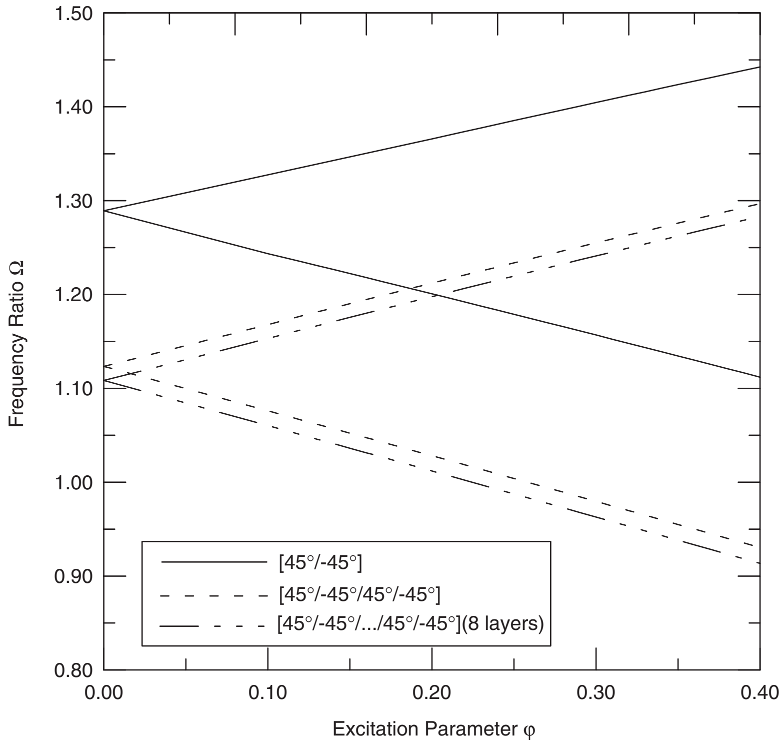

Fig. 9. The effect of the number of layers on the principal instability regions for the clamped rectangular angle-ply laminated plates.

$$Y_j(\eta) = C_j\left(\cosh\left(\frac{\lambda_j\eta}{b}\right) - \cos\left(\frac{\lambda_j\eta}{b}\right)\right) - \left(\sinh\left(\frac{\lambda_j\eta}{b}\right) - \sin\left(\frac{\lambda_j\eta}{b}\right)\right), \tag{16}$$

$$C_k = (\sinh\lambda_k - \sin\lambda_k)/(\cosh\lambda_k - \cos\lambda_k)$$

and λ_k is satisfied by the characteristic equation

$$1 - \cosh\lambda_k\cos\lambda_k = 0. \tag{17}$$

The external in-plane forces acting on the plate are uniformly distributed along two opposite edges ($\xi = 0$ and a), but the other two edges ($\eta = 0$ and b) are stress free. These in-plane boundary conditions can be expressed as follows:

$$\begin{aligned} &\phi_{,\eta\eta} = -(N_0 + N_t\cos\lambda t) \quad \text{and} \quad \phi_{,\xi\eta} = 0 \text{ at } \xi = 0, a, \\ &\phi_{,\xi\xi} = 0 \quad \text{and} \quad \phi_{,\xi\eta} = 0 \quad \text{at } \eta = 0, b. \end{aligned} \tag{18}$$

The stress function, ϕ, satisfying the in-plane conditions, is also assumed to be

$$\phi = \sum_p \sum_q h\phi_{pq}X_p(\xi)Y_q(\eta) - \frac{\eta^2}{2}(N_0 + N_t\cos\lambda t), \tag{19}$$

where X_i, Y_j are beam eignfunction as defined in Eqs. (15) and (16).

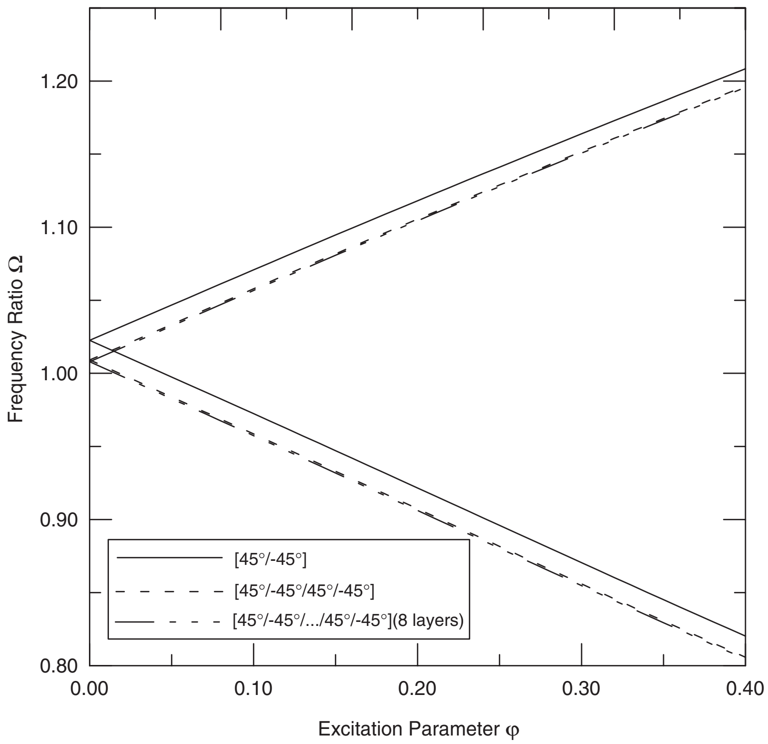

Fig. 10. The effect of the number of layers on the principal instability regions for the simply supported skew ($\alpha = 60^\circ$) angle-ply laminated plates.

Eqs. (12) and (19), are substituted into Eqs. (8) and (9). The first resulting equation is multiplied by $X_i(\xi)Y_j(\eta)$ and the second by $\psi_i(\xi)Z_j(\eta)$. Integrating both equations with respect to ξ from 0 to a and η from 0 to b, the results are

$$h[G_{ij}^{mn}]\phi_{mn} - h[H_{ij}^{pq}]W_{pq} + h^2[V_{ij}^{pqrs}]W_{pq}W_{rs} = 0, \tag{20}$$

$$\begin{aligned}\rho h^2[S_{ij}^{pq}]W_{pq,tt} + h[P_{ij}^{pq}]W_{pq} + h[Q_{ij}^{mn}]\phi_{mn} - h^2\phi_{mn}^{\mathrm{T}}[R_{ij}^{mnpq}]W_{pq}\\ + h^2[T_{ij}^{pq}](N_0 + N_t\cos\lambda t)W_{pq} = 0,\end{aligned} \tag{21}$$

where

$$[G_{ij}^{mn}] = \int_0^b \int_0^a L_2(X_mY_n)X_iY_j\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$[H_{ij}^{mn}] = \int_0^b \int_0^a L_3(\psi_mZ_n)X_iY_j\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$[P_{ij}^{mn}] = \int_0^b \int_0^a L_1(\psi_mZ_n)\psi_iZ_j\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

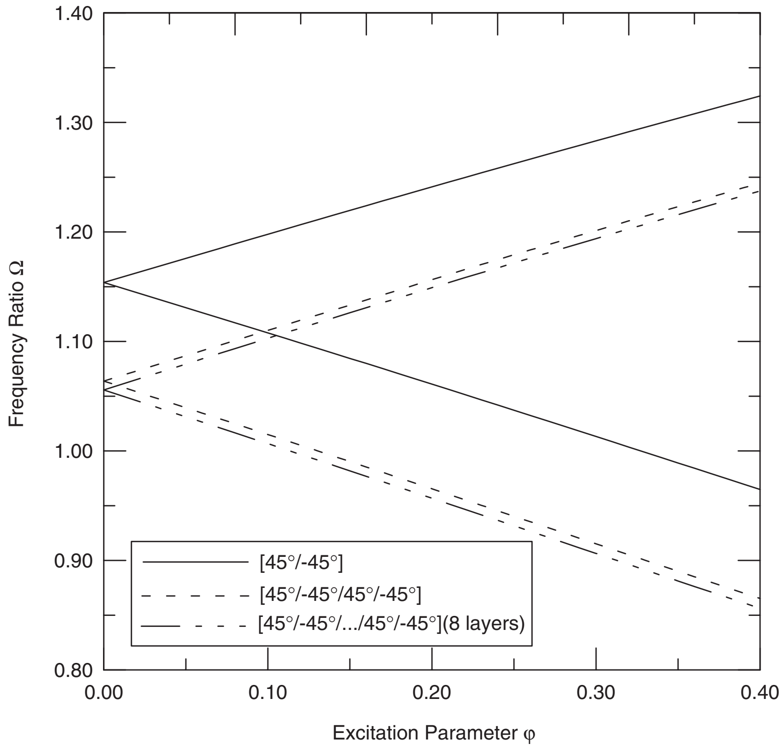

Fig. 11. The effect of the number of layers on the principal instability regions for the clamped skew ($\alpha = 60^{\circ}$) angle-ply laminated plates.

$$[Q_{ij}^{mn}] = \int_0^b \int_0^a L_3(X_m Y_n)\psi_i Z_j \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$[S_{ij}^{mn}] = \int_0^b \int_0^a \psi_m Z_n \psi_i Z_j \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$[T_{ij}^{mn}] = \int_0^b \int_0^a \csc^2\alpha \psi''_m Z_n \psi_i Z_i \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$[V_{ij}^{mnpq}] = \int_0^b \int_0^a (\psi''_m z_n \psi_p z''_q - \psi'_m z'_n \psi'_p z'_q) X_i Y_j \,\mathrm{d}x\,\mathrm{d}y,$$

$$[R_{ij}^{mnpq}] = \int_0^b \int_0^a (X_m Y''_n \psi''_p Z_p + X''_m Y_n \psi_p Z_p - 2 X_m Y_n \psi_p Z_q)\psi_i Z_j \,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Eq. (20) can be rewritten as

$$\phi_{mn} = [G_{ij}^{mn}]^{-1}(H_{ij}^{pq} W_{pq} - h V_{ij}^{pqrs} W_{pq} W_{rs}). \tag{22}$$

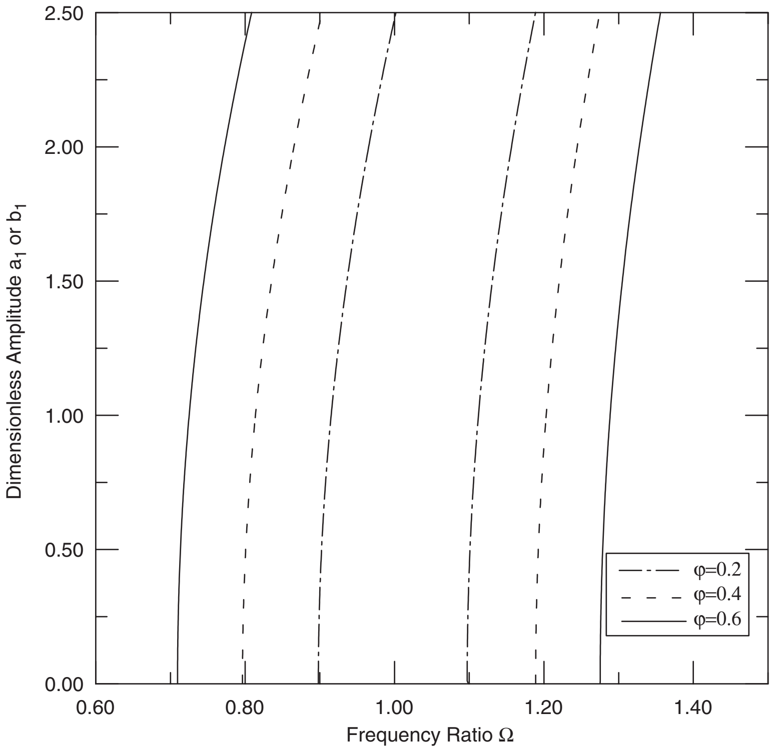

Fig. 12. Nonlinear effect of amplitude on the principal instability regions for the simply supported rectangular angle-ply [45°/−45°/45°/−45°] laminated plates. Flags are on the stable sides.

Substituting Eq. (22) into Eq. (21) leads to a system of ordinary differential equations as follows:

$$\begin{aligned}\frac{\mathrm{d}^2 W_{ij}}{\mathrm{d}t^2} &+ [\alpha_{0ij}^{pq}]W_{pq} + [\alpha_{1ij}^{pq}]\cos\lambda t\, W_{pq} + [\beta_{ij}^{pqrs}]W_{qp}W_{rs} \\ &+ [\gamma_{ij}^{pqrstu}]W_{pq}W_{rs}W_{tu} = 0,\end{aligned} \tag{23}$$

where

$$[\alpha_{0ij}^{pq}] = [S_{ij}^{pq}]^{-1}([P_{ij}^{pq}] + [Q_{ij}^{mn}][G_{ij}^{mn}]^{-1}[H_{ij}^{pq}] + N_0[T_{ij}^{pq}])/(\rho h),$$

$$[\alpha_{1ij}^{pq}] = N_t[S_{ij}^{pq}]^{-1}[T_{ij}^{pq}]/(\rho h),$$

$$[\beta_{ij}^{pqrs}] = [S_{ij}^{pq}]^{-1}(-[Q_{ij}^{mn}][G_{ij}^{mn}]^{-1}[V_{ij}^{pqrs}] - [G_{ij}^{mn}]^{-1}[H_{ij}^{ps}][R_{ij}^{mnrs}])/\rho,$$

$$[\gamma_{ij}^{pqrstu}] = h[S_{ij}^{pq}]^{-1}([G_{ij}^{mn}]^{-1}[V_{ij}^{pqrs}][R_{ij}^{mntu}])/\rho,$$

In this study, the only first terms of W_{pq} ($p = 1$, $q = 1$) and ϕ_{mn} ($m = 1$, $n = 1$) are considered, the modal Eq. (23) becomes

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} + (\alpha_0 + \alpha_1\cos\lambda t)W + \beta W^2 + \gamma W^3 = 0. \tag{24}$$

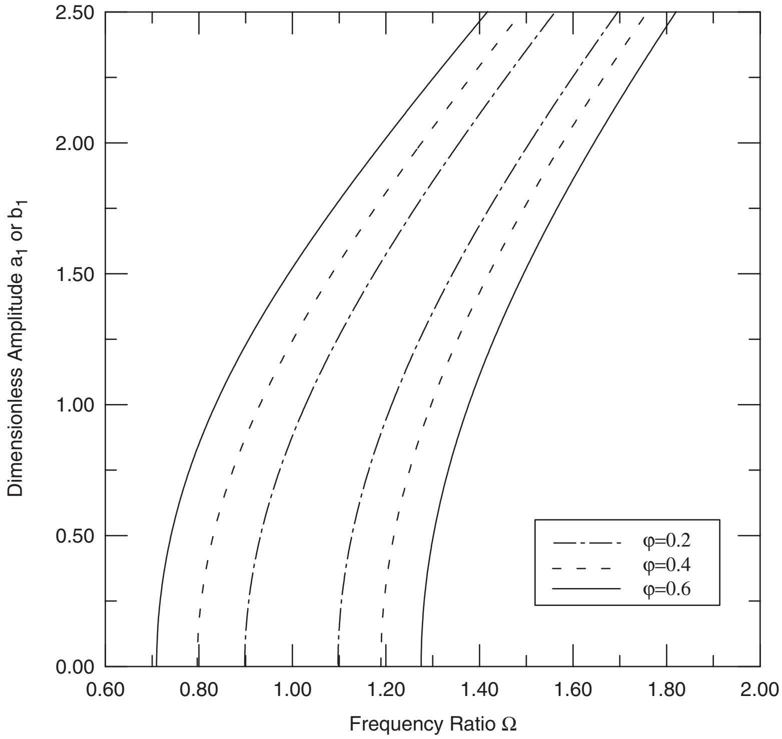

Fig. 13. Nonlinear effect of amplitude on the principal instability regions for the clamped rectangular angle-ply [45°/−45°/45°/−45°] laminated plates.

Eq. (24) can be written as

$$\frac{d^2W}{dt^2} + \omega_L^2(1 + 2\varphi\cos\lambda t)W + \beta W^2 + \gamma W^3 = 0, \tag{25}$$

where $\omega_L^2 = \alpha_0$, $2\varphi = \alpha_1/\alpha_0$.

In this study, the new parameters are defined as $\omega = \lambda/2$, $\Omega = \omega/\omega_L$, $\tau = \omega t$, $k_1 = \beta/\omega_L^2$ and $k_2 = \gamma/\omega_L^2$, Eq. (25) becomes

$$\Omega^2\frac{d^2W}{d\tau^2} + (1 + 2\varphi\cos 2\tau)W + k_1W^2 + k_2W^3 = 0. \tag{26}$$

3.2. Incremental harmonic balance (IHB) method

The IHB method has been successfully applied to various types of nonlinear dynamic problems and discussed in a number of papers: for example, Refs. [21–24]. The procedure of the IHB method for seeking periodic solutions is generally divided into two steps. In the first step, small increments are added to the current solution of equation. The current state of vibration corresponding to a point (Ω_0, φ_0) on instability boundary is denoted by W_0. A neighboring state

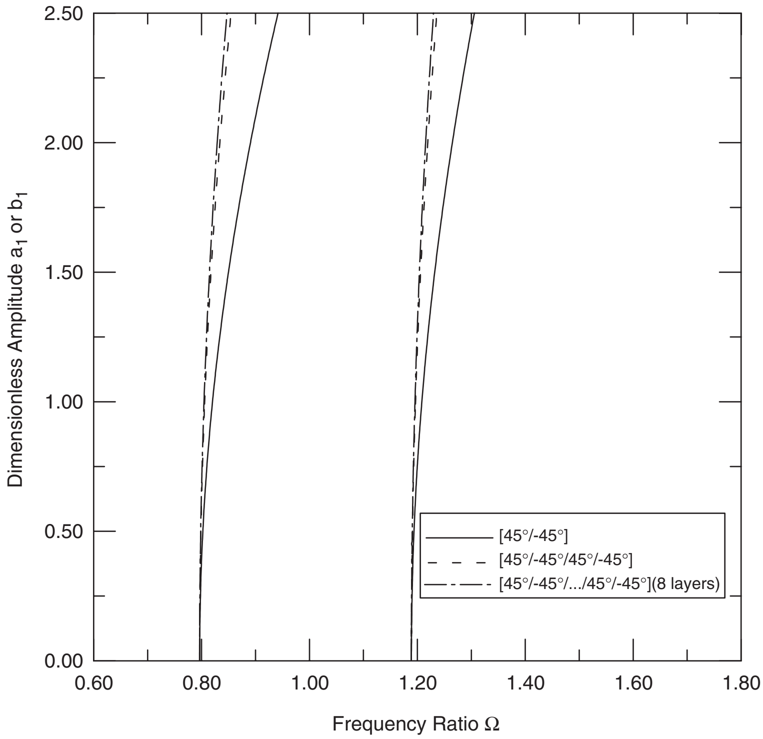

Fig. 14. Nonlinear effect of amplitude on the principal instability regions for the simply supported skew ($\alpha = 60^\circ$) angle-ply laminated plates.

is reached through a parameter incrementation:

$$\varphi = \varphi_0 + \Delta\varphi, \quad \Omega = \Omega_0 + \Delta\Omega, \quad W = W_0 + \Delta W. \tag{27}$$

Substituting the expansions (27) into Eq. (26) and neglecting the nonlinear terms of $\Delta\varphi$, $\Delta\Omega$, ΔW, a linearized incremental equation is obtained

$$\begin{aligned}\Omega_0^2 \Delta\ddot{W} + (1 + 2\varphi\cos 2\tau)\Delta W + 2k_1 W_0 \Delta W + 3k_2 W_0^2 \Delta W \\ = R + 2\Delta\varphi W_0 \cos 2\tau - 2\Delta\Omega\Omega_0 \ddot{W}_0,\end{aligned} \tag{28a}$$

$$R = -[\Omega_0^2 \ddot{W}_0 + (1 + 2\varphi_0 \cos 2\tau) W_0 + k_1 W_0^2 + k_2 W_0^3]. \tag{28b}$$

As mentioned in Refs. [21,22], although Eq. (28a) is linear, it has variable coefficients, and thus is difficult to solve. Hence, an approximate solution will be obtained by assuming a periodic solution and using the Galerkin's method, which is the second step. The approximate functions W_0 and ΔW can be expanded as a truncated Fourier series

$$W_0(\tau) = \sum_{k=1,3,\ldots}^{2N-1} (a_k \sin k\tau + b_k \cos k\tau), \tag{29}$$

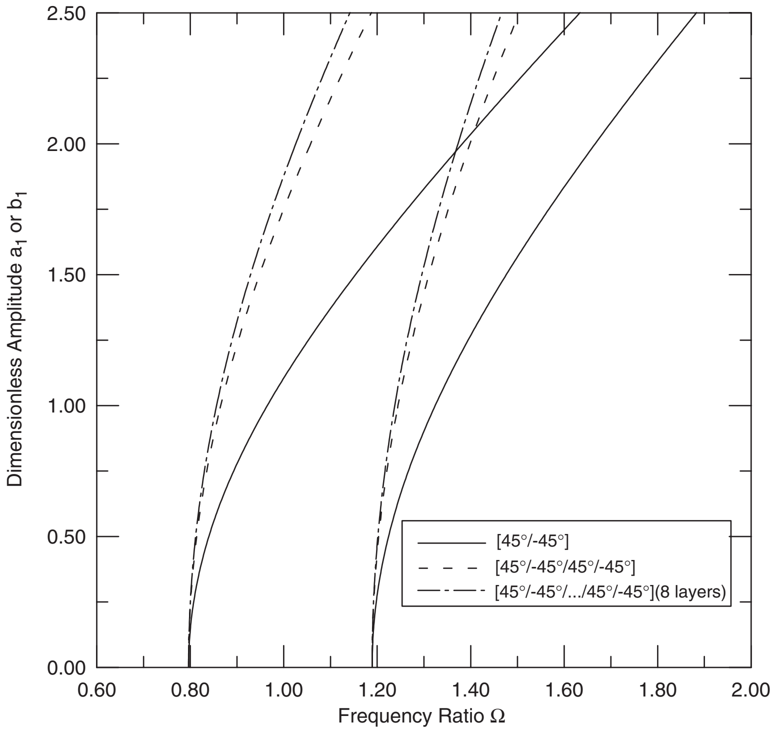

Fig. 15. Nonlinear effect of amplitude on the principal instability regions for the clamped supported skew ($\alpha = 60^\circ$) angle-ply laminated plates.

$$\Delta W(\tau) = \sum_{k=1,3,\ldots}^{2N-1} (\Delta a_k \sin k\tau + \Delta b_k \cos k\tau) \tag{30}$$

for solution with period 2π in terms of τ. N is the number of temporal terms for calculation. The coefficients a_k and b_k are defined as the dimensionless amplitude of the system [22]. By considering Eq. (29), one may note that the vibration of the system is represented in an equivalent manner by the values of the coefficients. The choice of the coefficients a_k and b_k will be discussed later in this section. Substituting Eqs. (29) and (30) into Eq. (28a), and applying Galerkin procedure, a set of linear equations can be obtained as follows:

$$[C]\{\Delta a\} = \{R\} + \Delta\psi\{P\} + \Delta\Omega\{Q\}, \tag{31}$$

where $[C]$ is the matrix for the Fourier coefficients and $\{\Delta a\}$ is a vector consisting of Fourier coefficients Δa_k or Δb_k, for example: $\{\Delta a\}^{\mathrm{T}} = \{\Delta a_1, \Delta a_3, \Delta a_5, \ldots\}$. The vectors $\{R\}$, $\{P\}$ and $\{Q\}$ can be calculated corresponding to Eq. (28b), the second and third right-hand side terms of Eq. (31). The IHB method needs only to treat a series of linear algebraic equations. Because only the relative value as of the coefficients in Eqs. (29) and (30) are required, one of them is possible to prescribe as a unity reference constant with its corresponding increment in $\{\Delta a\}$ set to zero, for example, $a_1 = 1.0$ and $\Delta a_1 = 0$. Hence, the summation is on $k = 1, 3, 5, \ldots, 2N-1$ for the principal region of instability, corresponding to a solution of period 2π.

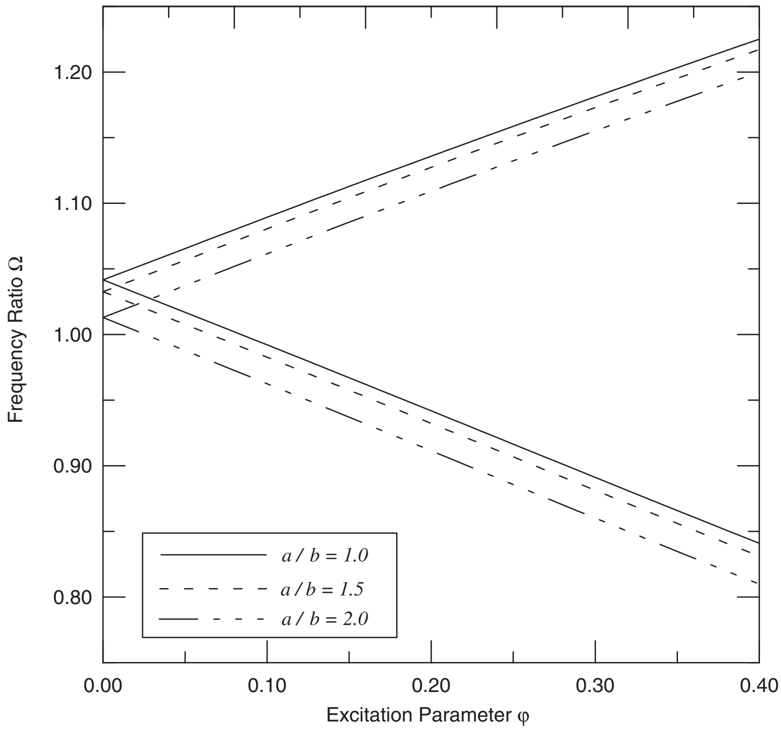

Fig. 16. The effect of aspect ratio (a/b) on the principal instability regions for the simply supported rectangular angle-ply [45°/−45°] laminated plates.

From Eq. (31), a linear system of $2N$ equations with $2N+2$ unknowns Δa, $\Delta\varphi$ and $\Delta\Omega$ has to be solved at each incremental step. Hence, it is necessary to add two constraints among Δa, $\Delta\varphi$ and $\Delta\Omega$. The first constraint is $\Delta a_1 = 0$ and the second constraint either $\Delta\varphi = 0$ or $\Delta\Omega = 0$. If $\Delta\varphi = 0$, then φ is said to be an active increment [21,22], i.e., the boundary curve is obtained by incrementing φ. This is so-called φ-incrementation. Similarly, if $\Delta\Omega = 0$, then Ω is an active increment (Ω-incrementation). Therefore, only $2N$ equations are needed for solving the problem. These two constraints have been discussed in detail in Refs. [21,22]. When $\Delta\varphi = 0$ has been used as the second constraint, then Eq. (31) can be written as

$$\begin{aligned} C_{12}^{\mathrm{T}}\Delta a &= R_1 + \Delta\Omega Q_1, \\ [C_3]\Delta a &= R + \Delta\Omega Q, \end{aligned} \tag{32}$$

where

$$[C] = \left\{ \begin{matrix} C_1 & \vdots & C_{12}^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ C_{21} & \vdots & [C_3] \end{matrix} \right\}, \quad \Delta a = \begin{bmatrix} 0 \\ \Delta a \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ R \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 \\ Q \end{bmatrix},$$

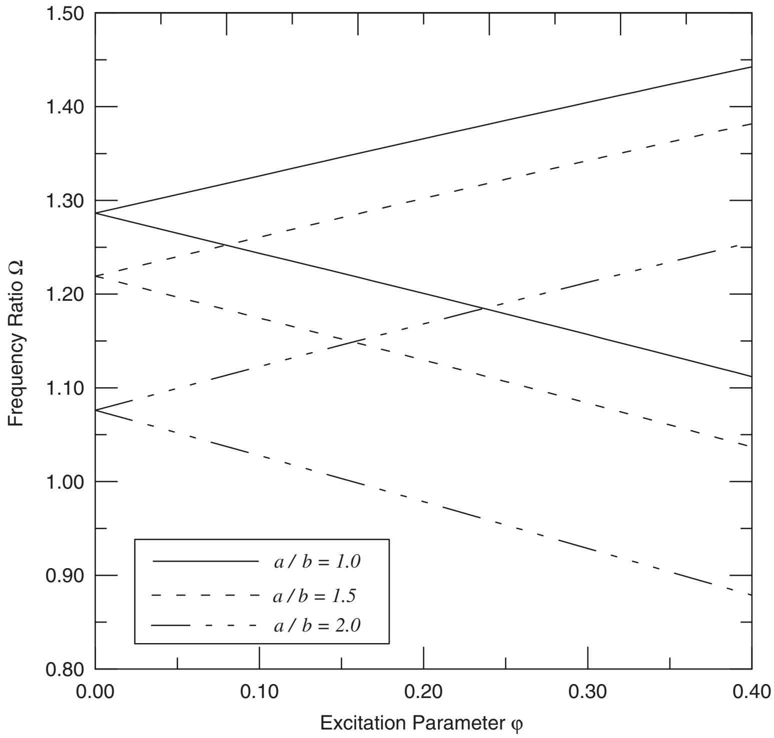

Fig. 17. The effect of aspect ratio (a/b) on the principal instability regions for the clamped supported rectangular angle-ply [45°/−45°] laminated plates.

T denotes a transpose. Solving Eq. (32), $\Delta\Omega$ is obtained

$$\Delta\Omega = \frac{C_{12}^{\mathrm{T}}[C_3]^{-1}R - R_1}{P_1 - C_{12}^{\mathrm{T}}[C_3]^{-1}Q}. \tag{33}$$

Then, Δa is calculated as

$$\Delta a = [C_3]^{-1}\{R + \Delta\Omega Q\}. \tag{34}$$

Substituting the solution into Eq. (28b), R is the corrective term, and becomes zero when the solution is reached.

4. Results and discussions

In this study, the general solutions of an arbitrarily laminated thin skew ($\alpha = 60^\circ$) and rectangular ($\alpha = 90^\circ$) plate with all four edges simply supported (S–S–S–S) and four edges clamped (C–C–C–C) have been developed. Solutions of isotropic, angle-ply and arbitrary laminated plates are considered. The dynamic instability regions and frequency ratios are determined and compared with other results which are available in the literature.

4.1. Isotropic plates

The material constants for an isotropic material are applied as $E_1/E_2 = 1.0$, $G_{12}/E_2 = 1/2(1+v)$, $v = 0.3$, $\rho = 1190\,\mathrm{kg\,m^{-3}}$, $a/b = 1.0$, $a/h = 250$, $N_0 = 0.004\,\mathrm{N\,m^{-1}}$ and $N_t = -0.04\,\mathrm{N\,m^{-1}}$. The parametric instability region associated with an isotropic clamped plate as $a_1 = 0.01$ and $b_1 = 0.01$ has been calculated and shown in Fig. 2 wherein the comparison with the region obtained by Srinivasan and Chellapandi [8] using the finite element method is made. These results are found to be in good agreement. The principal dynamic instability regions for simply supported and clamped rectangular ($\alpha = 90^\circ$) plates are shown in Figs. 3 and 4. While $\alpha = 60^\circ$, the principal dynamic instability regions for simply supported and clamped skew plates are shown in Figs. 5 and 6. In Figs. 3–6, the amplitude affects the regions of dynamic instability because of large amplitude plate theory.

4.2. Angle-ply laminated plates

The material constants for laminate material (typical of graphite epoxy) are considered as $E_1/E_2 = 40$, $G_{12}/E_2 = 0.6$, $G_{13}/E_2 = G_{23}/E_2 = 0.5$, $v_{12} = 0.25$, $\rho = 1578\,\mathrm{kg\,m^{-3}}$, $a/b = 1.0$, $a/h = 100$, $N_0 = 3.24\,\mathrm{N\,m^{-1}}$ and $N_t = -0.9\,\mathrm{N\,m^{-1}}$. All the laminates are assumed to be of the same thickness and material properties.

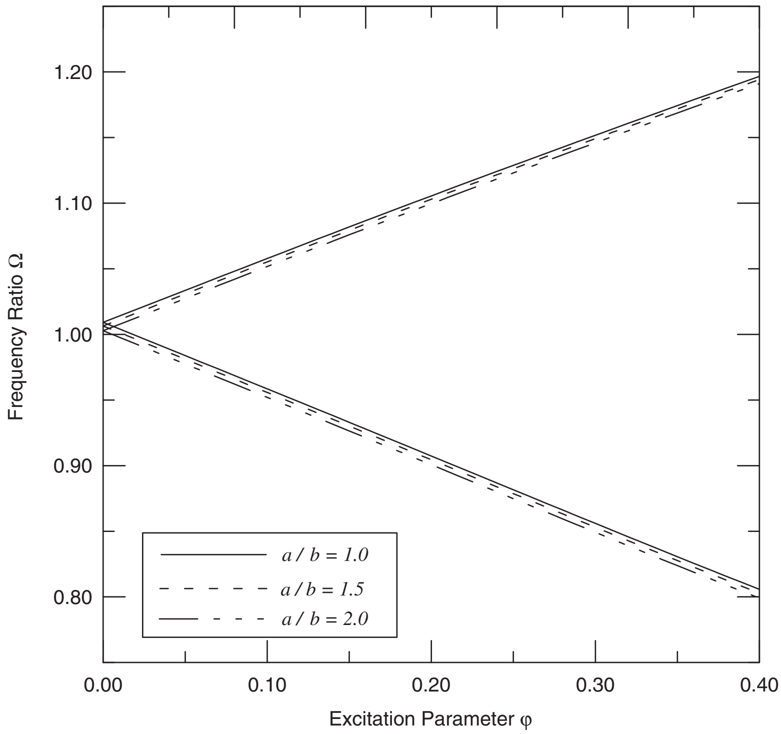

Fig. 18. The effect of aspect ratio (a/b) on the principal instability regions for the simply supported skew ($\alpha = 60^\circ$) angle-ply 45°/−45°/45°/−45° laminated plates.

The accuracy in determining the regions of parametric instability depends on the natural frequency and the critical buckling load of the plate. The dimensionless natural frequency is

$$\bar{\Omega} = \omega_L \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}, \tag{35}$$

where ω_L can be obtained from Eq. (25).

The dimensionless fundamental frequencies of simply supported square angle-ply laminated plates have been determined. The fundamental frequency of the plate with aspect ratios and length-to-thickness ratio obtained in this investigation have been compared in Tables 1 and 2 with the results by Reddy [6], Bert and Chen [25]. Table 3 shows a comparison of dimensionless frequencies, for various layers laminate square plate obtained by various investigators. In Table 4, the dimensionless frequencies of simply supported antisymmetric angle-ply laminates are presented. In the studies of Reddy [6], Bert and Chen [25], Chen and Yang [7], Reddy and Palaninathan [19], the shear deformation and rotary inertia effects have been considered. In this study, $a/h = 50$ and 100 for thin plates are used and the shear deformation and rotary inertia effects are neglected. In Tables 1–4, the dimensionless natural frequencies obtained in present work approach the results of the other research. Therefore, the governing equations of thin plate and the analytical method considered in this study are reasonable.

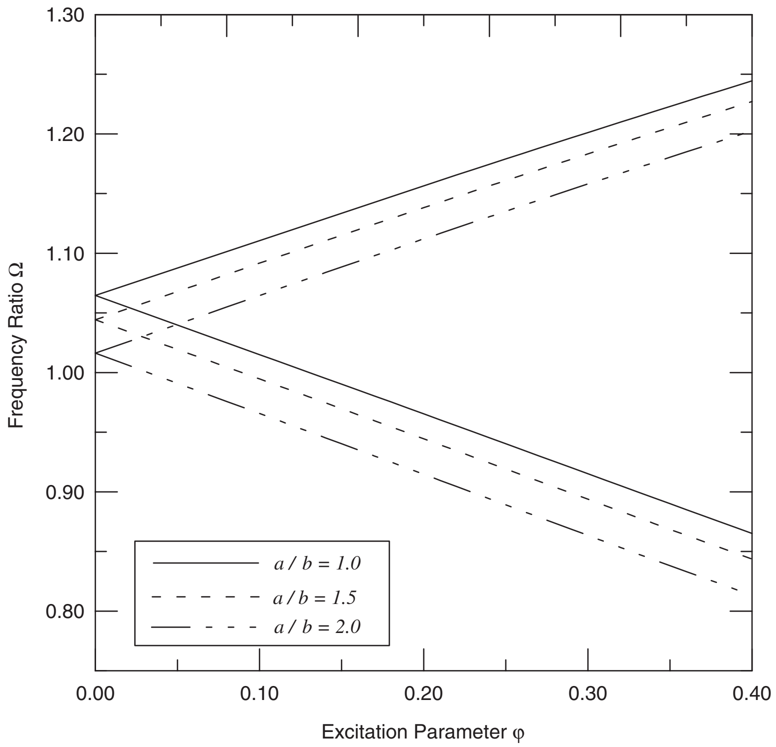

Fig. 19. The effect of aspect ratio (a/b) on the principal instability regions for the clamped skew ($\alpha = 60^\circ$) angle-ply $45^\circ/-45^\circ/45^\circ/-45^\circ$ laminated plates.

In this paper, the rectangular and skew laminated plates have been calculated for [45°/−45°], [45°/−45°/45°/−45°] and [45°/−45°/…/45°/−45°] (8 layers) with different boundary conditions. The parametric instability region associated with a four-layer angle-ply clamped laminated plate as $a_1 = 0.01$ and $b_1 = 0.01$ has been calculated and shown in Fig. 7, wherein it is compared with the region obtained by Ref. [8]. These two regions of dynamic instability are closed. In Figs. 8 and 9, the principal instability regions for the antisymmetric two-layer ([45°/−45°]) rectangular and skew plates are shown.

The effect of the number of layers and the principal regions of dynamic instability as $a_1 = 1.0$ and $b_1 = 1.0$ for the simply supported and clamped rectangular ($\alpha = 90°$) angle-ply laminated plates are shown in Figs. 8 and 9. While $\alpha = 60°$, the principal regions of dynamic instability for the simply supported and clamped skew angle-ply laminated plates are shown in Figs. 10 and 11. In Figs. 8–11, the effect of stacking sequence on the regions of dynamic instability for the same plate thickness is obvious.

Using the φ-incrementation procedure, the nonlinear effects of amplitude on the principal instability region for the simply supported and clamped rectangular angle-ply [45°/−45°/−45°/−45°] laminated plates are shown in Figs. 12 and 13, respectively. While $\varphi = 0.4$, the effects of varying the number of layers on the principal region of instability associated with the different boundary conditions for the skew angle of plate ($\alpha = 60°$) are shown in Figs. 14 and 15. As it can

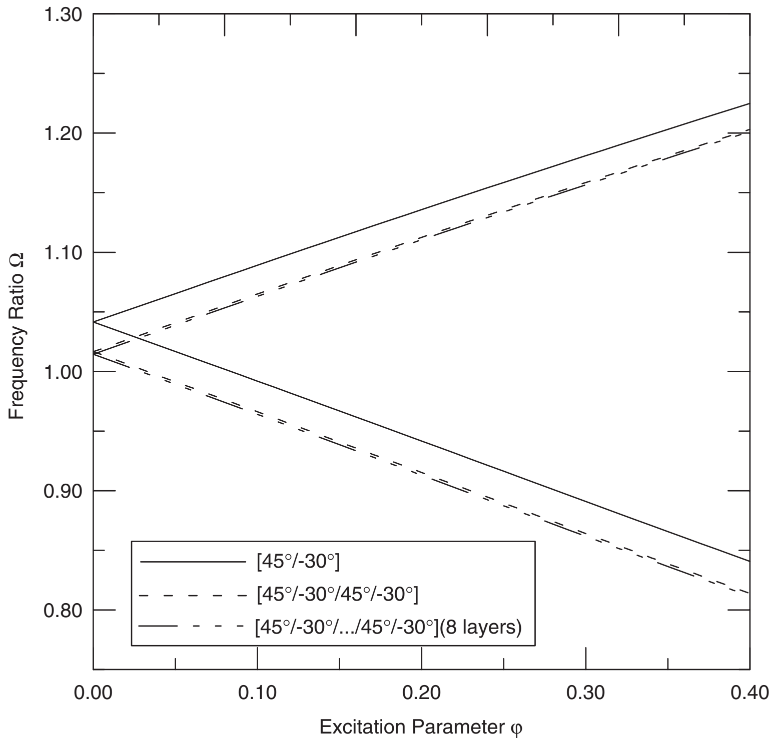

Fig. 20. The effect of the number of layers on the principal instability regions for the simply supported rectangular arbitrarily laminated plates.

be seen, an increase in amplitude a_1 or b_1 has the beneficial effect of increasing the relative value of frequency ratio Ω. The beneficial effect is more pronounced for a clamped condition than for a simply supported condition. In addition, it is easy to observe that an increase in the number of layers decreases the frequency ratio Ω.

The effects of aspect ratio (a/b) on the principal instability region associated with the different boundary conditions and the skew angle of the plate are illustrated in Figs. 16–19. In these figures, the increase in the aspect ratio of plate increases the dimensionless fundamental frequency but decreases the frequency ratio Ω. The results also indicate that, when the skew angle of the plate is taken into account, the frequency ratio Ω corresponding to the specified aspect ratio is lower than the rectangular plate. Further, the effect of decreasing the relative value of frequency ratio Ω for the clamped condition is more obvious than the simply supported condition.

4.3. Arbitrarily laminated plates

The material constants for laminate material (typical of graphite epoxy) shown in Section 4.2 and $a_1 = 1.0$ and $b_1 = 1.0$ are considered. Figs. 20 and 21 show the principal regions of dynamic instability and the effect of the number of layers for the simply supported and clamped rectangular ($\alpha = 90^\circ$) arbitrarily laminated plates consisting of two, four and eight layers oriented

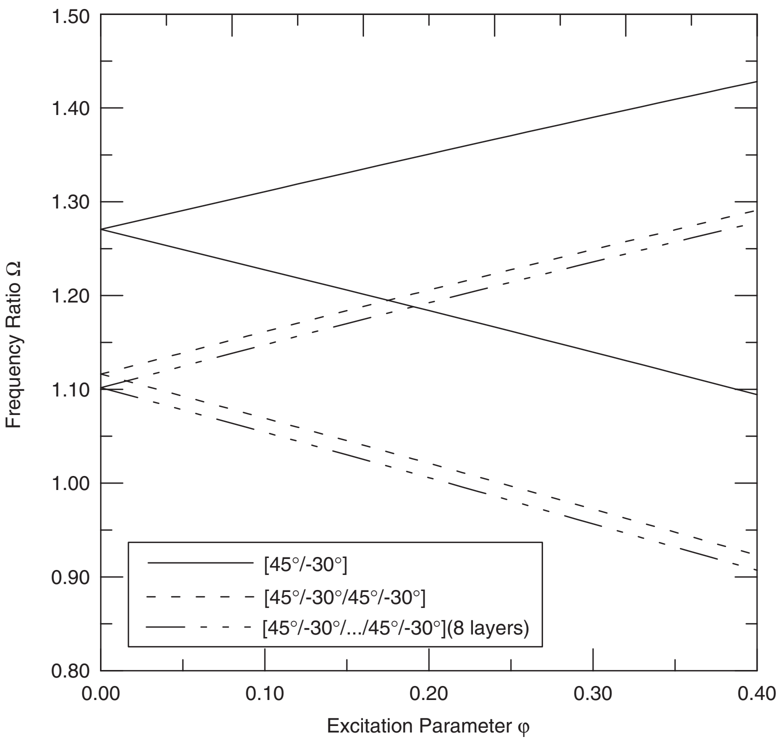

Fig. 21. The effect of the number of layers on the principal instability regions for the clamped rectangular arbitrarily laminated plates.

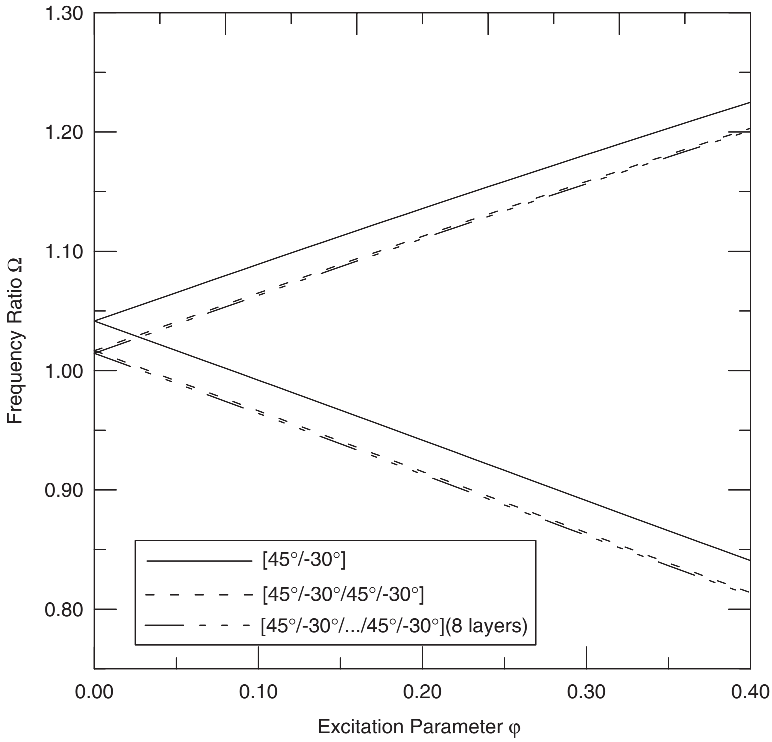

Fig. 22. The effect of the number of layers on the principal instability regions for the simply supported skew ($\alpha = 60^\circ$) arbitrarily laminated plates.

at $45^\circ/-30^\circ$, $45^\circ/-30^\circ/45^\circ/-30^\circ$, $45^\circ/-30^\circ/\ldots/45^\circ/-30^\circ$ (8 layers). In Figs. 22 and 23, results for the skew ($\alpha = 60^\circ$) arbitrarily laminate plates are shown.

At the same values of the load factor φ, the difference between the upper and lower bounds of the primary instability region can be used as an instability measures to study the influence of the other parameters. A similar phenomenon is also observed in Section 4.2, the increase in the number of layers increases the natural frequency, but reduces the parameter Ω value. As can be seen, the all four edges clamped plate has a significant increase in the stability and nonlinear natural frequency. It seems to be reasonable that the more rigid boundary condition has greater stability.

5. Conclusions

Based on von Karman's equations and incremental harmonic balance method, the region of dynamic instability is determined for isotropic, angle-ply and arbitrarily laminated plates. The rectangular and skew plates with simply supported and clamped boundary conditions are considered in this study. The influence of amplitude on the region of dynamic instability is significant. The effects of materials, lamination orientations are considerable.

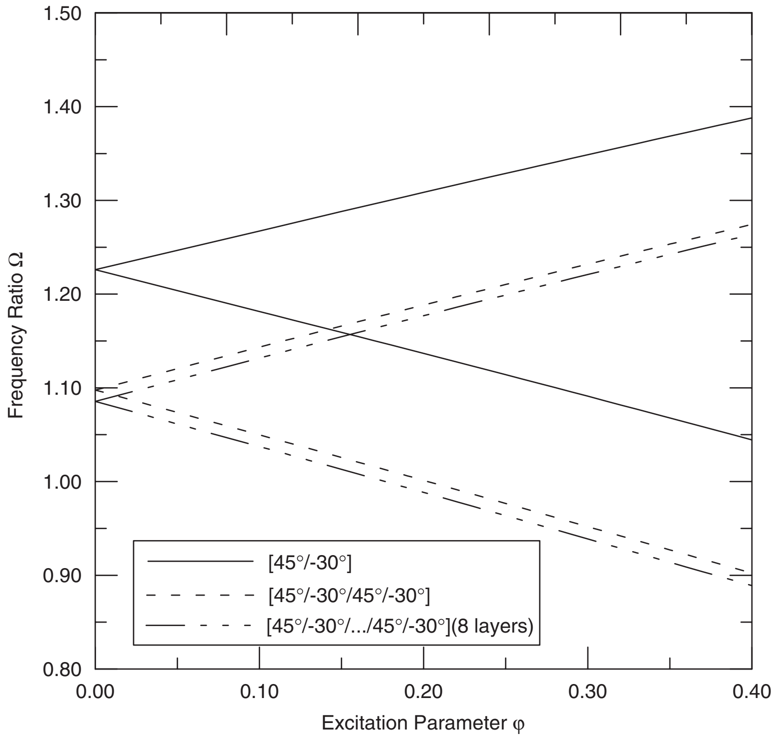

Fig. 23. The effect of the number of layers on the principal instability regions for the clamped skew ($\alpha = 60^\circ$) arbitrarily laminated plates.

The effects of the boundary condition, stacking sequence of lamination, in-plane load, aspect ratio, amplitude and skew angle are noted. The results from this study can be summarized as follows:

(1) The stability behavior of the system is determined by the several parameters, such as amplitude (a_1 and b_1), load factor φ, frequency ratio Ω, aspect ratio (a/b), the boundary condition, number of the layers, stacking sequence and the skew angle of plate.
(2) When the increase in the number of layers increases the natural frequency, but reduces the frequency ratio Ω, the structure is more stable.
(3) For the same values of the parameters, the increase in the value of a_1 or b_1 increases the nonlinear frequency of the system and the frequency ratio Ω.
(4) The increase in the load factor φ increases the instability region.
(5) When the skew angle of the plate is taken into account, the frequency ratio Ω corresponding to the specified aspect ratio is lower than the rectangular plate.
(6) The increase in the aspect ratio of plate increases the dimensionless fundamental frequency but decreases the frequency ratio Ω.
(7) The more rigid boundary condition has greater stability.

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